Announcements

1) Colin Adams, CB 1030,
2-3, general
audience talk

Back to Lemma

Let $E \subseteq \mathbb{R}^n$ be a convex, open set (in $\|\cdot\|_2$). Then if $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable $\forall x \in E$ and satisfies $\sup_{x \in E} \|f'(x)\| \leq M$ for some $M \geq 0$, then $\forall x, y \in E$,

$$\boxed{\|f(x) - f(y)\|_2 \leq M \|x - y\|_2}$$

back to proof:

For $a, b \in E$ we had

defined $\varphi(t) = ta + (1-t)b$,

$0 \leq t \leq 1$.

We get

$$g : [0,1] \to \mathbb{R}^n$$

$$g(t) = f(\varphi(t)).$$

It suffices to show

$$\|g(0) - g(1)\|_2 \leq M \|a - b\|_2 .$$

Now define

$h(t) : [0,1] \to \mathbb{R}$

by

$$h(t) = (g(1) - g(0)) \cdot g(t)$$

Calculate $h'(t)$.

$$h'(t) = (g(1) - g(0))' \cdot g(t) + (g(1) - g(0)) \cdot g'(t)$$

$$= (g(1) - g(0)) \cdot g'(t)$$

since $g(1) - g(0)$ is constant.

Using Cauchy-Schwarz,

$$|h'(t)| = |(g(1)-g(0))\cdot g'(t)|$$

$$\leq \|g(1)-g(0)\|_2 \, \|g'(t)\|_2$$

$$\leq \|g(1)-g(0)\|_2 \, M \|a-b\|_2$$

But by the Mean Value Theorem, there exists a c, $c \in (0,1)$, with

$$h(1)-h(0) = h'(c)$$

$$\Rightarrow |h(1)-h(0)|$$

$$= |h'(c)|$$

$$\leq \|g(0)-g(1)\|_2 \cdot M \cdot \|b-a\|_2$$

But $h(1)-h(0)$

$$= (g(1)-g(0))\cdot g(1)$$

$$- (g(1)-g(0))\cdot g(0)$$

$$= (g(1)-g(0))\cdot (g(1)-g(0)).$$

This is $\|g(1)-g(0)\|_2^2$

$$= \|g(0)-g(1)\|_2^2$$

$$= \|f(a)-f(b)\|_2^2$$

so we get

$$\|f(a)-f(b)\|_2^2 = \|g(0)-g(1)\|_2^2$$

$$\leq \|g(0)-g(1)\|_2 M \|b-a\|_2$$

$$= \|f(a)-f(b)\|_2 M \|b-a\|_2$$

so dividing by $\|f(a)-f(b)\|_2$,

We get

$$\|f(a)-f(b)\|_2 \leq M \|a-b\|_2.$$

Since $a, b \in E$ were arbitrary, this gives the result. $\square$

Theorem: (contractions)

Let (X, d) be a complete metric space and let $\varphi : X \to X$ satisfying: $\exists\, c,\ 0 < c < 1,$

$$d(\varphi(x), \varphi(y)) \leq c\, d(x, y)$$

(φ is a contraction) $\forall x, y \in X$.

Then $\exists\ x_0 \in X,\ \varphi(x_0) = x_0$

i.e. φ fixes x_0.

Proof: Choose $x \in X$.

Consider the sequence

$$\left(\varphi^n(x)\right)_{n=1}^{\infty}$$

where $\varphi^n = \underbrace{\varphi \circ \varphi \circ \cdots \circ \varphi}_{n \text{ times}}$.

We will show that

$\left(\varphi^n(x)\right)_{n=1}^{\infty}$ is Cauchy.

Consider

$$d(\varphi^n(x), \varphi^m(x)).$$

Suppose $n \geq m$.

Then by repeated applications of the contraction property, we get

$$d(\varphi^n(x), \varphi^m(x)) \leq C^m d(\varphi^{n-m}(x), x)$$

Suppose $n \geq m \geq N$

Then $d(\varphi^n(x), \varphi^m(x))$

$$\leq \sum_{k=m}^{n-1} d(\varphi^{k+1}(x), \varphi^k(x))$$

$$\leq \sum_{k=m}^{n-1} c^k d(\varphi(x), x)$$

$$\leq d(\varphi(x), x) \sum_{k=N}^{\infty} c^k$$

$$= d(\varphi(x), x) \frac{c^N}{1-c}$$

If $d(\varphi(x), x) = 0$, then $x = \varphi(x)$ and we are done.

If $d(\varphi(x), x) \neq 0$, then

$$0 < c < 1 \Rightarrow \lim_{k \to \infty} c^k = 0,$$ so

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \quad c^N < \frac{\varepsilon(1-c)}{d(x, \varphi(x))}.$$

Then $\forall n, m \geq N$,

$$d(\varphi^n(x), \varphi^m(x)) \leq d(\varphi(x), x) \frac{c^N}{1-c} < \varepsilon.$$

The sequence is Cauchy, therefore since $\underline{X}$ is complete, it admits a limit x_0.

$$\lim_{n \to \infty} \varphi^n(x) = x_0$$

$$\Rightarrow \lim_{n \to \infty} \varphi^{n+1}(x) = \varphi(x_0)$$

by applying φ (note φ continuous).

But $\lim_{n \to \infty} \varphi^{n+1}(x) = \lim_{n \to \infty} \varphi^n(x) = x_0$,

so $x_0 = \varphi(x_0)$

Moreover, x_0 is unique since

Since if $\varphi(y_0) = y_0$,

$$d(x_0, y_0) = d(\varphi(x_0), \varphi(y_0))$$
$$\leq c\, d(x_0, y_0)$$

$\Rightarrow d(y_0, x_0) = 0$, so

$x_0 = y_0$. $\square$

Inverse Function Theorem

Let $f : E \to \mathbb{R}^n$ where $E \subseteq \mathbb{R}^n$ is open.

Suppose $\exists$ an $a \in E$ with $f'(a)$ invertible. If f is differentiable on E, and f' is continuous on E

a) $\exists\ \varepsilon > 0$ such that

f is 1-1 on $B(a,\varepsilon) = U$,

and moreover, $f(U)$ is open.

b) If we let $g = f^{-1} : f(U) \to U$,

then g is differentiable

on $f(U)$, and

$$g'(y) = \left(f'(f^{-1}(y))\right)^{-1}$$

$\forall\ y \in f(U)$.